Existence of infinitely many solutions for the fractional Schrödinger- Maxwell equations *†

Zhongli Wei[a,b]

School of Sciences, Shandong Jianzhu University,

Jinan, Shandong, 250101, People's Republic of China

[b] School of Mathematics, Shandong University,

Jinan, Shandong 250100, People's Republic of China. .

Abstract

In this paper, by using variational methods and critical point theory, we shall mainly study the existence of infinitely many solutions for the following fractional Schrödinger-Maxwell equations

$$(-\Delta)^{\alpha}u + V(x)u + \phi u = f(x,u), \text{in } \mathbb{R}^3,$$

$$(-\triangle)^{\alpha}\phi = K_\alpha u^2 \ \text{ in } \ \mathbb{R}^3$$

where $\alpha \in (0,1]$, $K_\alpha = \dfrac{\pi^{-\alpha}\Gamma(\alpha)}{\pi^{-(3-2\alpha)/2}\Gamma((3-2\alpha)/2)}$, $(-\Delta)^{\alpha}$ stands for the fractional Laplacian. Under some more assumptions on f, we get infinitely many solutions for the system.

Key words Fractional Laplacian, Schrödinger-Maxwell equations, infinitely many solutions.

2000 MR. Subject Classification 35B40, 35B45, 35J55, 35J60, 47J30.

1 Introduction and the Main Result

In this paper, we study the fractional Schrödinger-Maxwell equations

$$(-\Delta)^{\alpha}u + V(x)u + \phi u = f(x,u), \text{in} \quad \mathbb{R}^3 \ , \tag{1.1}$$

$$(-\triangle)^{\alpha}\phi = K_\alpha u^2 \ \text{ in } \ \mathbb{R}^3 \tag{1.2}$$

*E-mail address: jnwzl32@163.com (Z.L. Wei).

†Research supported by the NSF of Shandong Province (ZR2013AM009).

where $u, \phi : \mathbb{R}^3 \to \mathbb{R}, f : \mathbb{R}^3 \times \to \mathbb{R}$, $\alpha \in (0, 1]$, $K_\alpha = \dfrac{\pi^{-\alpha}\Gamma(\alpha)}{\pi^{-(3-2\alpha)/2}\Gamma((3-2\alpha)/2)}$, $(-\triangle)^\alpha$ stands for the fractional Laplacian. Here the fractional Laplacian $(-\triangle)^\alpha$ with $\alpha \in (0, 1]$ of a function $\phi : \mathbb{R}^3 \to \mathbb{R}$ is defined by:

$$\mathcal{F}((-\triangle)^\alpha \phi)(\xi) = |\xi|^{2\alpha}\mathcal{F}(\phi)(\xi), \ \forall \alpha \in (0, 1],$$

where $\mathcal{F}$ is the Fourier transform, i.e.,

$$\mathcal{F}(\phi)(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \exp\{-2\pi i \xi \cdot x\}\phi(x)\mathrm{d}x.$$

If ϕ is smooth enough, $(-\Delta)^\alpha$ can also be computed by the following singular integral :

$$(-\triangle)^\alpha \phi(x) = c_{3,\alpha}\mathrm{P.V.} \int_{\mathbb{R}^3} \frac{\phi(x) - \phi(y)}{|x - y|^{3+2\alpha}}\mathrm{d}y.$$

Here P.V. is the principal value and $c_{3,\alpha}$ is a normalization constant. Such a system (1.1) is called Schrödinger-Maxwell equations or Schrödinger-Poisson equations which is obtained while looking for existence of standing waves for the fractional nonlinear Schrödinger equations interacting with an unknown electrostatic field. For a more physical background of system (1.1), we refer the reader to [1, 2] and the references therein.

When $\alpha = 1$, system (1.1) was first introduced by Benci and Fortunato in [1], and it has been widely studied by many authors; The case $V \equiv 1$ or being radially symmetric, has been studied under various conditions on f in [3]-[9]; When $V(x)$ is not a constant, the existence of infinitely many large solutions for (1.1) has been considered in [10]-[14] via the fountain theorem (cf. [15, 16].)

In system (1.1), we assume the following hypotheses on potential V and nonlinear term f :

($\mathbb{V}$) $V \in C(\mathbb{R}^3, \mathbb{R})$, $\inf_{x \in \mathbb{R}^3} V(x) \geq a_1 > 0$, where a_1 is a positive constant. Moreover, $\lim\limits_{|x| \to \infty} V(x) = +\infty$.

($\mathbb{H}_1$) $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, and there exist $c_1, c_2 > 0$, $p \in (4, 2^*_\alpha)$ such that

$$|f(x, u)| \leq c_1|u| + c_2|u|^{p-1}, \ \ \forall \ x \in \mathbb{R}^3, \ u \in \mathbb{R},$$

where, $2^*_\alpha = \frac{6}{3-2\alpha}$, $\alpha > \frac{3}{4}$, $f(x, u)u \geq 0$ for $u \geq 0$.

($\mathbb{H}_2$) $\lim_{|u| \to \infty} \frac{F(x,u)}{u^4} = +\infty$ uniformly for $x \in \mathbb{R}^3$, here $F(x, u) = \int_0^u f(x, t)\,\mathrm{d}t$.

($\mathbb{H}_3$) Let $G(x, u) = \frac{1}{4}f(x, u)u - F(x, u)$, there exist $a_0 > 0$, and $g(x) \geq 0$ such that $\int_{\mathbb{R}^3} g(x)\mathrm{d}x < +\infty$, $G(x, u) \geq -a_0 g(x)$, $\quad \forall \ (x, u) \in \mathbb{R}^3 \times \ \mathbb{R}$.

($\mathbb{H}_4$) $f(x, -u) = -f(x, u) \ \forall \ x \in \mathbb{R}^3, \ u \in \mathbb{R}$.

Now, we are ready to state the main result of this paper.

Theorem 1.1. *Assume that* $(\mathbb{V})$ *and* $(\mathbb{H}_1)-(\mathbb{H}_4)$ *satisfy. Then system* (1.1) *possesses infinitely many nontrivial solutions.*

Remark 1.1. (i) : *There are functions* f *satisfying the assumptions* $(\mathbb{H}_1)-(\mathbb{H}_4)$, *for example* (1) : $f(x,u) = 4u^3\ln(u^2+1)+\frac{2u^5}{(u^2+1)}$, *then* $a_0 = 0$, $(\mathbb{H}_3)$ *is satisfied;* (2) : $f(x,u) = e^{-\sum_{i=1}^{3}|x_i|}u+|u|^{p-2}u,\ p\in(4,2_\alpha^*),\ \alpha>\frac{3}{4}$, *then* $a_0 = \frac{r_0^2}{4}, g(x) = e^{-\sum_{i=1}^{3}|x_i|}, r_0 = \left(\frac{p}{p-4}\right)^{1/(p-2)}+1$, $(\mathbb{H}_3)$ *is satisfied.*

(ii) : *the assumption* $(\mathbb{H}_3)$ *is weaker than the assumptions* (f_4) *in paper* [12] *and* $(f3')$ *in paper* [14].

2 Variational settings and preliminary results

Now, let′s introduce some notations. For any $1\le r<\infty, L^r(\mathbb{R}^3)$ is the usual Lebesgue space with the norm

$$\|u\|_{L^r} = \left(\int_{\mathbb{R}^3}|u(x)|^r\,\mathrm{d}x\right)^{\frac{1}{r}}.$$

The fractional order Sobolev space:

$$H^\alpha(\mathbb{R}^3) = \left\{u\in L^2(\mathbb{R}^3):\ \int_{\mathbb{R}^3}(|\xi|^{2\alpha}\hat{u}^2+\hat{u}^2)\ \mathrm{d}\xi<\infty\right\},$$

where $\hat{u} = \mathcal{F}(u)$, The norm is defined by

$$\|u\|_{H^\alpha(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3}(|\xi|^{2\alpha}\hat{u}^2+\hat{u}^2)\ \mathrm{d}\xi\right)^{\frac{1}{2}}.$$

The spaces $D^\alpha(\mathbb{R}^3)$ is defined as the completion of $C_0^\infty(\mathbb{R}^3)$ under the norms

$$\|u\|_{D^\alpha(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3}(|\xi|^{2\alpha}\hat{u}^2\mathrm{d}\xi\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^3}|(-\Delta)^{\alpha/2}u(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}.$$

Note that, by Plancherel's theorem we have $\|u\|_2 = \|\hat{u}\|_2$, and

$$\begin{aligned}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}u(x)|^2\mathrm{d}x &= \int_{\mathbb{R}^3}((\widehat{-\Delta)^{\frac{\alpha}{2}}u}(\xi))^2\mathrm{d}\xi = \int_{\mathbb{R}^3}(|\xi|^\alpha\hat{u}(\xi))^2\mathrm{d}\xi\\ &= \int_{\mathbb{R}^3}|\xi|^{2\alpha}\hat{u}^2\mathrm{d}\xi<\infty,\ \forall u\in H^\alpha(\mathbb{R}^3).\end{aligned}$$

It follows that

$$\|u\|_{H^\alpha(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{\alpha}{2}}u(x)|^2+u^2)\mathrm{d}x\right)^{\frac{1}{2}}.$$

In our problem, we work in the space defined by

$$E := \left\{u\in H^\alpha(\mathbb{R}^3)\ \Big|\ \left(\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{\alpha}{2}}u(x)|^2+V(x)u^2)\mathrm{d}x\right)^{\frac{1}{2}}<\infty\right\}. \tag{2.1}$$

Thus, E is a Hilbert space with the inner product

$$(u,v)_E := \int_{\mathbb{R}^3} \big((-\Delta)^{\frac{\alpha}{2}} u(x) \cdot (-\Delta)^{\frac{\alpha}{2}} v(x) + V(x)uv\big)\, \mathrm{d}x.$$

and its norm is $\|u\| = (u,u)^{\frac{1}{2}}$. Obviously, under the assumptions $(\mathbb{V})$, $\|u\|_E \equiv \|u\|_{H^\alpha}$.

Lemma 2.1 (see [17] Lemma 2.2 and [18]). *$H^\alpha(\mathbb{R}^3)$ is continuously embedded into $L^p(\mathbb{R}^3)$ for $p \in [2, 2_\alpha^*]$; and compactly embedded into $L^p_{loc}(\mathbb{R}^N)$ for $p \in [2, 2_\alpha^*)$ where $2_\alpha^* = \dfrac{6}{3-2\alpha}$. Therefore, there exists a positive constant C_p such that*

$$\|u\|_p \le C_p \|u\|_{H^\alpha(\mathbb{R}^3)}.$$

Lemma 2.2 (see [19]). *Under the assumption $(\mathbb{V})$, the embedding E is compactly embedded into $L^p(\mathbb{R}^3)$ for $p \in [2, 2_\alpha^*)$.*

Lemma 2.3 (see [20]). *For $1 < p < \infty$ and $0 < \alpha < N/p$, we have*

$$\|u\|_{L^{\frac{pN}{N-p\alpha}}(\mathbb{R}^N)} \le B \|(-\Delta)^{\alpha/2} u\|_{L^p(\mathbb{R}^N)} \tag{2.2}$$

with best constant

$$B = 2^{-\alpha} \pi^{-\alpha/2} \frac{\Gamma((N-\alpha)/2)}{\Gamma((N+\alpha)/2)} \left(\frac{\Gamma((N)}{\Gamma(N/2)}\right)^{\alpha/N}.$$

Lemma 2.4. *For any $u \in H^\alpha(\mathbb{R}^N)$ and for any $h \in D^{-\alpha}(\mathbb{R}^N)$, there exists a unique solution $\phi = ((-\Delta)^\alpha + u^2)^{-1} h \in D^\alpha(\mathbb{R}^N)$ of the equation*

$$(-\Delta)^\alpha \phi + u^2 \phi = h,$$

(being $D^{-\alpha}(\mathbb{R}^N)$ the dual space of $D^\alpha(\mathbb{R}^N)$). Moreover, for every $u \in H^\alpha(\mathbb{R}^N)$ and for every $h, g \in D^{-\alpha}(\mathbb{R}^N)$,

$$\langle h, \big((-\Delta)^\alpha + u^2\big)^{-1} g\rangle = \langle g, \big((-\Delta)^\alpha + u^2\big)^{-1} h\rangle \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $D^{-\alpha}(\mathbb{R}^N)$ and $D^\alpha(\mathbb{R}^N)$.

Proof. If $u \in H^\alpha(\mathbb{R}^N)$, then by Hölder inequality and (2.2)

$$\int_{\mathbb{R}^N} u^2 \phi^2 \mathrm{d}x \le \|u\|_{2p}^2 \|\phi\|_{2q}^2 \le B^2 \|u\|_{2p}^2 \|\phi\|_{D^\alpha}^2, \tag{2.4}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $q = \frac{N}{N-2\alpha}$, $2q = 2_\alpha^*$. Thus $\left(\int |(-\Delta)^{\alpha/2}\phi|^2 + \int u^2\phi^2\right)^{1/2}$ is a norm in $D^\alpha(\mathbb{R}^N)$ equivalent to $\|\phi\|_{D^\alpha}$. Hence, by the application of Lax-Milgram Lemma, we

obtain the existence part. For every $u \in H^{\alpha}(\mathbb{R}^N)$ and for every $h, g \in D^{-\alpha}(\mathbb{R}^N)$, we have $\phi_g = ((-\Delta)^{\alpha} + u^2)^{-1} g$, $\phi_h = ((-\Delta)^{\alpha} + u^2)^{-1} h$. Hence,

$$\begin{aligned}
&\langle h, ((-\Delta)^{\alpha} + u^2)^{-1} g\rangle = \int h \left((-\Delta)^{\alpha} + u^2\right)^{-1} g \mathrm{d}x \\
&= \int h\phi_g \mathrm{d}x = \int \left((-\Delta)^{\alpha} + u^2\right) \phi_h \phi_g \mathrm{d}x \\
&= \int \left((-\Delta)^{\alpha}\phi_h + u^2\phi_h\right) \phi_g \mathrm{d}x = \int \left((-\Delta)^{\alpha}\phi_g + u^2\phi_g\right) \phi_h \mathrm{d}x \\
&= \int g\phi_h \mathrm{d}x = \int g \left((-\Delta)^{\alpha} + u^2\right)^{-1} h \mathrm{d}x = \langle g, \left((-\Delta)^{\alpha} + u^2\right)^{-1} h\rangle.
\end{aligned}$$

So, we get (2.3). □

Lemma 2.5 (see [21]). *Let f be a function in $C_0^{\infty}(\mathbb{R}^N)$ and let $0 < \alpha < n$. Then, with*

$$c_{\alpha} \doteq \pi^{-\alpha/2}\Gamma(-\alpha/2), \tag{2.5}$$

$$c_{\alpha}(\xi^{-\alpha}\widehat{f}(\xi))^{\vee}(x) = c_{n-\alpha}\int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)\mathrm{d}y. \tag{2.6}$$

Lemma 2.6. *For every $u \in H^{\alpha}$ there exists a unique $\phi = \phi(u) \in D^{\alpha}$ which solves equation* (1.2). *Furthermore, $\phi(u)$ is given by*

$$\phi(u)(x) = \int_{\mathbb{R}^3} |x-y|^{2\alpha-3} u^2(y)\mathrm{d}y. \tag{2.7}$$

As a consequence, the map $\Phi:\ u \in H^{\alpha} \mapsto \phi(u) \in D^{\alpha}$ is of class C^1 and

$$[\Phi(u)]'(v)(x) = 2\int_{\mathbb{R}^3} |x-y|^{2\alpha-3} u(y)v(y)\mathrm{d}y, \quad \forall u, v \in H^{\alpha}. \tag{2.8}$$

Proof. The existence and uniqueness part follows by Lemma 2.4. By Lemma 2.5 and the Fourier transform of equation (1.2), the representation formula (2.7) holds for $u \in C_0^{\infty}(\mathbb{R}^3)$; by density it can be extended for any $u \in H^{\alpha}$. The representation formula (2.8) is obvious. □

System (1.1) and (1.2) are the Euler-Lagrange equations corresponding to the functional $J : H^{\alpha}(\mathbb{R}^3) \times D^{\alpha}(\mathbb{R}^3) \to \mathbb{R}$ is

$$J(u,\phi) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|(-\Delta)^{\frac{\alpha}{2}} u(x)|^2 + V(x)u^2 - \frac{1}{2}|(-\Delta)^{\frac{\alpha}{2}}\phi(x)|^2 + K_{\alpha}\phi u^2\right) \mathrm{d}x - \int_{\mathbb{R}^3} F(x,u)\mathrm{d}x,$$

where $F(x,t) = \int_0^t f(x,s)\mathrm{d}s,\ t \in \mathbb{R}$.

Evidently, the action functional J belongs to $C^1(H^\alpha(\mathbb{R}^3)\times D^\alpha(\mathbb{R}^3),\mathbb{R})$ and the partial derivatives in (u,ϕ) are given, for $\xi\in H^\alpha(\mathbb{R}^3)$ and $\eta\in D^\alpha(\mathbb{R}^3)$, by

$$\left\langle \frac{\partial J}{\partial u}(u,\phi),\xi\right\rangle = \int_{\mathbb{R}^3}\left((-\Delta)^{\frac{\alpha}{2}}u(x)(-\Delta)^{\frac{\alpha}{2}}\xi(x)+V(x)u\xi+K_\alpha\phi u\xi\right)\mathrm{d}x-\int_{\mathbb{R}^3}f(x,u)\xi(x)\mathrm{d}x,$$

$$\left\langle \frac{\partial J}{\partial \phi}(u,\phi),\eta\right\rangle = \frac{1}{2}\int_{\mathbb{R}^3}\left(-(-\Delta)^{\frac{\alpha}{2}}\phi(x)(-\Delta)^{\frac{\alpha}{2}}\eta(x)+K_\alpha u^2\eta\right)\mathrm{d}x.$$

Thus, we have the following result:

Proposition 2.1. *The pair (u,ϕ) is a weak solution of system (1.1) and (1.2) if and only if it is a critical point of J in $H^\alpha(\mathbb{R}^3)\times D^\alpha(\mathbb{R}^3)$.*

So, we can consider the functional $J: H^\alpha(\mathbb{R}^3)\to\mathbb{R}$ defined by $J(u)=J(u,\phi(u))$. After multiplying (1.2) by $\phi(u)$ and integration by parts, we obtain

$$\int_{\mathbb{R}^3}|(-\triangle)^{\alpha/2}\phi(u)|^2\mathrm{d}x=K_\alpha\int_{\mathbb{R}^3}\phi(u)u^2\mathrm{d}x.$$

Therefore, the reduced functional takes the form

$$J(u)=\frac{1}{2}\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{\alpha}{2}}u(x)|^2+V(x)u^2)\mathrm{d}x+\frac{1}{4}K_\alpha\int_{\mathbb{R}^3}u^2\phi(u)\mathrm{d}x-\int_{\mathbb{R}^3}F(x,u)\mathrm{d}x. \quad (2.9)$$

Lemma 2.7. *Assume that there exist $c_1,c_2>0$ and $p>1$ such that*

$$|f(s)|=c_1|s|+c_2|s|^{p-1},\quad \forall s\in\mathbb{R}. \quad (2.10)$$

Then the following statements are equivalent:

i) $(u,\phi)\in(H^\alpha\cap L^p)\times D^\alpha$ is a solution of the system (1.1) − (1.2);

ii) $u\in H^\alpha\cap L^p$ is a critical point of J and $\phi=\phi(u)$.

Proof. By the assumption (2.10), the Nemitsky operator $u\in H^\alpha\cap L^p\mapsto F(x,u)\in L^1$ is of class C^1. Hence, by Lemma 2.6, for every $u,v\in H^\alpha$

$$\begin{aligned} J'(u)[v] &= \int_{\mathbb{R}^3}(-\Delta)^{\frac{\alpha}{2}}u(x)(-\Delta)^{\frac{\alpha}{2}}v(x)\mathrm{d}x+\int_{\mathbb{R}^3}V(x)uv\mathrm{d}x\\ &\quad+\frac{1}{2}K_\alpha\int_{\mathbb{R}^3}uv\int_{\mathbb{R}^3}|x-y|^{2\alpha-3}u^2(y)\mathrm{d}y\mathrm{d}x\\ &\quad+\frac{1}{2}K_\alpha\int_{\mathbb{R}^3}u^2\int_{\mathbb{R}^3}|x-y|^{2\alpha-3}u(y)v(y)\mathrm{d}y\mathrm{d}x-\int_{\mathbb{R}^3}f(x,u)v\mathrm{d}x\\ &= \int_{\mathbb{R}^3}(-\Delta)^{\frac{\alpha}{2}}u(x)(-\Delta)^{\frac{\alpha}{2}}v(x)\mathrm{d}x+\int_{\mathbb{R}^3}V(x)uv\mathrm{d}x\\ &\quad+K_\alpha\int_{\mathbb{R}^3}uv\phi(u)\mathrm{d}x-\int_{\mathbb{R}^3}f(x,u)v\mathrm{d}x. \end{aligned}$$

By Fubini-Tonelli's Theorem, we can obtain the conclusion. □

If $1 \le p < \infty$ and $a, b \ge 0$, then

$$(a+b)^p \le 2^{p-1}(a^p + b^p). \tag{2.11}$$

From (1.2) and (2.2), for any $u \in E$ using Hölder inequality we have

$$\|\phi(u)\|_{D^\alpha}^2 = K_\alpha \int_{\mathbb{R}^3} \phi(u) u^2 \mathrm{d}x \le K_\alpha \|\phi(u)\|_q \|u\|_{2p}^2 \le C\|\phi(u)\|_{D^\alpha}\|u\|_{2p}^2.$$

where $\frac{1}{p} + \frac{1}{q} = 1,\ q = 2_\alpha^* = \frac{6}{3-2\alpha},\ \alpha > \frac{3}{4}$. Here and subsequently, C denotes an universal positive constant. This and lemma 2.2 implies that

$$\|\phi(u)\|_{D^\alpha} \le C\|u\|_{2p}^2 \le C\|u\|_E^2, \tag{2.12}$$

$$\int_{\mathbb{R}^3} \phi(u)u^2 \mathrm{d}x \le C\|u\|_{2p}^4 \le C\|u\|_E^4. \tag{2.13}$$

Lemma 2.8. *Assume that a sequence $\{u_n\} \subset E$, $u_n \rightharpoonup u$ in E as $n \to \infty$ and $\{u_n\}$ be a bounded sequence. Then*

$$\left| \int_{\mathbb{R}^3} (\phi(u_n)u_n - \phi(u)u)(u_n - u)\mathrm{d}x \right| \to 0, \quad \text{as} \quad n \to \infty.$$

Proof. Let $\{u_n\}$ be a sequence satisfying the assumptions $u_n \rightharpoonup u$ in E as $n \to \infty$ and $\{u_n\}$ is bounded. Lemma 2.2 implies that $u_n \to u$ in $L^r(\mathbb{R}^3)$, where $2 \le r < 2_\alpha^*$, and $u_n \to u$ for a.e. $x \in \mathbb{R}^3$. Hence $\sup_{n\in\mathbb{N}} \|u_n\|_r < \infty$ and $\|u\|_r$ is finite. By Hölder inequality, (2.11), (2.12) and (2.4)

$$\begin{aligned}
&\left| \int_{\mathbb{R}^3} (\phi(u_n)u_n - \phi(u)u)(u_n - u)\mathrm{d}x \right| \\
&\le \left(\int_{\mathbb{R}^3} (\phi(u_n)u_n - \phi(u)u)^2 \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} (u_n - u)^2 \mathrm{d}x \right)^{\frac{1}{2}} \\
&\le \left(2\int_{\mathbb{R}^3} (|\phi(u_n)u_n|^2 + |\phi(u)u|^2)\mathrm{d}x \right)^{\frac{1}{2}} \|u_n - u\|_2 \\
&\le C(\|u_n\|_E^6 + \|u\|_E^6)^{\frac{1}{2}} \|u_n - u\|_2 \to 0, \quad \text{as} \quad n \to \infty.
\end{aligned} \tag{2.14}$$

□

3 Proof of Theorem 1.1

We say that $J \in C^1(X, \mathbb{R})$ satisfies the $(C)_c$-condition if any sequence $\{u_n\}$ such that

$$J(u_n) \to c, \quad \|J'(u_n)\|(1 + \|u_n\|) \to 0$$

has a convergent subsequence, where X is a Banach space.

Lemma 3.1. *Assume that* $(\mathbb{V})$ *and* $(\mathbb{H}_1)-(\mathbb{H}_4)$ *satisfy. Then any sequence* $\{u_n\}\subset E$ *satisfying*

$$J(u_n)\to c>0,\quad \langle J'(u_n),u_n\rangle\to 0,$$

is bounded in E. *Moreover,* $\{u_n\}$ *contains a converge subsequence.*

Proof. To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\|\to\infty$ as $n\to\infty$. By $(\mathbb{H}_3)$ for sufficiently large $n\in\mathbb{N}$

$$\begin{aligned} c+1 &\geq J(u_n)-\frac{1}{4}\langle J'(u_n),u_n\rangle \\ &=\frac{1}{4}\|u_n\|^2+\int_{\mathbb{R}^3}G(x,u_n)\mathrm{d}x \\ &\geq \frac{1}{4}\|u_n\|^2-a_0\int_{\mathbb{R}^3}g(x)\mathrm{d}x\to+\infty. \end{aligned}$$

Thus $\sup_{n\in\mathbb{N}}\|u_n\|<\infty$. i.e. $\{u_n\}$ is a bounded sequence.

Now we shall prove $\{u_n\}$ contains a subsequence, without loss of generality, by Eberlein-Shmulyan theorem (see for instance in [22]), passing to a subsequence if necessary, there exists a $u\in E$ such that $u_n\rightharpoonup u$ in E, again by Lemma 2.2, $u_n\to u$ in $L^s(\mathbb{R}^3)$, for $2\leq s<2^*_\alpha$ and $u_n\to u$ a.e. $x\in\mathbb{R}^3$. By $(\mathbb{H}_1)$ and using Hölder inequality we have

$$\begin{aligned} &\left|\int_{\mathbb{R}^3}(f(x,u_n)-f(x,u))(u_n-u)\mathrm{d}x\right| \\ &\leq \int_{\mathbb{R}^3}\left|c_1(|u_n|+|u|)+c_2(|u_n|^{p-1}+|u|^{p-1})\right||u_n-u|\mathrm{d}x \\ &\leq c_1(\|u_n\|_2+\|u\|_2)\|u_n-u\|_2+c_2(\|u_n\|_p^{p-1}+\|u\|_p^{p-1})\|u_n-u\|_p \\ &\to 0,\ \text{as } n\to\infty. \end{aligned}$$

Since $J\in C^1(E)$, we have $J'(u_n)\to J'(u)$ in E^*. i.e.

$$\langle J'(u_n)-J'(u),u_n-u\rangle\to 0,\ \text{as } n\to\infty.$$

This together with Lemma 2.8 implies

$$\begin{aligned} \|u_n-u\|^2 &=\langle J'(u_n)-J'(u),u_n-u\rangle-K_\alpha\int_{\mathbb{R}^3}(\phi(u_n)u_n-\phi(u)u)(u_n-u)\mathrm{d}x \\ &\quad+\int_{\mathbb{R}^3}(f(x,u_n)-f(x,u))(u_n-u)\mathrm{d}x\to 0,\ \text{as } n\to\infty. \end{aligned}$$

That is $u_n\to u$ in E. □

Lemma 3.2. *Suppose that assumptions* $(\mathbb{V})$, $(\mathbb{H}_1)$ *and* $(\mathbb{H}_2)$ *satisfy, for any finite dimensional subspace* $\widetilde{E}\subset E$, *there holds*

$$J(u)\to-\infty,\quad \|u\|\to\infty,\quad u\in\widetilde{E}. \tag{3.1}$$

Proof. Arguing indirectly, assume that for some sequence $\{u_n\} \subset \widetilde{E}$ with $\|u_n\| \to \infty$, there is $M > 0$ such that $J(u_n) \geq -M$, $\forall\, n \in \mathbb{N}$. Set $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in E. Since $\dim \widetilde{E} < \infty$, then $v_n \to v \in \widetilde{E}$, $v_n(x) \to v(x)$ a.e. on $x \in \mathbb{R}^3$, and so $\|v\| = 1$. Let $\Omega := \{x \in \mathbb{R}^3 : v(x) \neq 0\}$, then $\mathrm{meas}(\Omega) > 0$ and for a.e. $x \in \Omega$, we have $\lim_{n\to\infty} |u_n(x)| \to \infty$. It follows from (2.9), (2.13) that

$$\lim_{n\to\infty} \frac{4\displaystyle\int_{\mathbb{R}^3} F(x, u_n)\mathrm{d}x}{\|u_n\|^4} = \lim_{n\to\infty} \frac{2\|u_n\|^2 + K_\alpha \displaystyle\int_{\mathbb{R}^3} \phi(u_n)u_n^2\mathrm{d}x - 4J(u_n)}{\|u_n\|^4} \leq C. \tag{3.2}$$

But by the non-negative of F, (($\mathbb{H}_2$) and Fadous Lemma, for large n we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{4\displaystyle\int_{\mathbb{R}^3} F(x, u_n)\mathrm{d}x}{\|u_n\|^4} \geq \lim_{n\to\infty} \int_\Omega \frac{4F(x, u_n)v_n^4}{u_n^4}\mathrm{d}x \\
&\geq \liminf_{n\to\infty} \int_\Omega \frac{4F(x, u_n)v_n^4}{u_n^4}\mathrm{d}x \\
&\geq \int_\Omega \liminf_{n\to\infty} \frac{F(x, u_n)v_n^4}{u_n^4}\mathrm{d}x \\
&= \int_\Omega \liminf_{n\to\infty} \frac{F(x, u_n)}{u_n^4}[\chi_\Omega(x)]v_n^4\mathrm{d}x \to \infty,\ n \to \infty.
\end{aligned}$$

This contradicts to (3.2). □

Corollary 3.1. *Under assumptions* ($\mathbb{V}$), ($\mathbb{H}_1$) *and* ($\mathbb{H}_2$), *for any finite dimensional subspace* $\widetilde{E} \subset E$, *there is* $R = R(\widetilde{E}) > 0$ *such that*

$$J(u) \leq 0, \quad \forall u \in \widetilde{E},\ \|u\| \geq R. \tag{3.3}$$

Let $\{e_j\}$ is an orthonomormal basis of E and define $X_j = \mathbb{R}e_j$,

$$Y_k = \oplus_{j=1}^k X_j, \quad Z_k = \oplus_{j=k+1}^\infty X_j, \quad k \in \mathbb{N}. \tag{3.4}$$

Lemma 3.3. *Under assumptions* ($\mathbb{V}$), *for* $2 \leq r < 2_\alpha^*$, *we have*

$$\beta_k(r) = \sup_{u\in Z_k, \|u\|=1} \|u\|_r \to 0, \quad k \to \infty. \tag{3.5}$$

Proof. Since the embedding from E into $L^r(\mathbb{R}^3)$ is compact, then Lemma 3.3 can be proved by a similar way as Lemma 3.8 in [15]. □

By Lemma 3.3, we can choose an integer $m \geq 1$ such that

$$\|u\|_2^2 \leq \frac{1}{2c_1}\|u\|^2, \quad \|u\|_p^p \leq \frac{p}{4c_2}\|u\|^p, \quad \forall u \in Z_m. \tag{3.6}$$

Lemma 3.4. *Suppose that assumptions* $(\mathbb{V})$ *and* $(\mathbb{H}_1)$ *are satisfied, there exist constants* $\rho, \delta > 0$ *such that* $J|_{\partial B_\rho \cap Z_m} \geq \delta > 0$.

Proof. By $(\mathbb{H}_1)$, we have

$$F(x,u) \leq \frac{c_1}{2}u^2 + \frac{c_2}{p}|u|^p, \ \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}.$$

Hence, by (2.9) and (3.6), we have

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}K_\alpha \int_{\mathbb{R}^3} \phi(u)u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(x,u)\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^3} F(x,u)\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|^2 - \frac{c_1}{2}\|u\|_2^2 - \frac{c_2}{p}|\|u\|_p^p \\ &\geq \frac{1}{4}(\|u\|^2 - \|u\|^p). \end{aligned}$$

Hence for any given $0 < \rho < 1$, let $\delta = \frac{1}{4}(\rho^2 - \rho^p)$, then $J|_{\partial B_\rho \cap Z_m} \geq \delta > 0$. This complete the proof. $\square$

Lemma 3.5 (see[23])**.** *Let* X *be an infinite dimensional Banach space,* $X = Y \oplus Z$*, where* Y *is finite dimensional. If* $J \in C^1(X, \mathbb{R})$ *satisfies* $(C)_c$*-condition for all* $c > 0$*, and*

(J1) $J(0) = 0, J(-u) = J(u)$ *for all* $u \in X$;

(J2) there exist constants $\rho, \delta > 0$ *such that* $J|_{\partial B_\rho \cap Z_m} \geq \delta > 0$;

(J3) for any finite dimensional subspace $\widetilde{E} \subset E$*, there is* $R = R(\widetilde{E}) > 0$ *such that* $J(u) \leq 0, \quad \forall u \in \widetilde{E} \backslash B_R$;

then J *possesses an unbounded sequence of critical values.*

Proof of Theorem 1.1. Let $X = E, Y = Y_m$ and $Z = Z_m$. By Lemmas 3.2, 3.4 and Corollary 3.1, all conditions of Lemma 3.5 are satisfied. Thus, problem (1.1) and (1.2) possesses infinitely many nontrivial solutions. $\square$

References

[1] T. D'Aprile, D. Mugnai, Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations, Proc. Roy. Soc. Edinburgh Sect. A **134** (2004), 893–906.

[2] V. Benci, D. Fortunato, An eigenvalue problem for the Schrödinger-Maxwell equations. Topol. Methods Nonl. Anal. **11** (1998) 283–293.

[3] A. Ambrosetti, D. Ruiz, Multiple bound states for the Schrödinger-Poisson problem. Commun. Contemp. Math. 10 (2008) 391-404.

[4] G.M. Coclite, A multiplicity result for the nonlinear SchrödingerMaxwell equations. Commun. Appl. Anal. 7 (2003) 417-423.

[5] T. D'Aprile, Non-radially symmetric solution of the nonlinear Schrödinger equation coupled with Maxwell equations. Adv. Nonlinear Stud. 2 (2002) 177-192.

[6] H. Kikuchi, On the existence of solution for elliptic system related to the Maxwell-Schrödinger equations. Nonlinear Anal. 27 (2007) 1445-1456.

[7] D. Ruiz, The Schrödinger-Possion equation under the effect of a nonlinear local term. J. Funct. Anal. 237 (2006) 655-674.

[8] L. Jeanjean, On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer type problem set on $\mathbb{R}^N$. Proc. Roy. Soc. Edinburgh Sect. A 129 (1999) 787-809.

[9] L. Zhao, F. Zhao, Positive solutions for Schrödinger-Poisson equations with a critical exponent, Nonlinear Anal. 70 (2009) 2150-2164.

[10] A. Azzollini, A. Pomponio, Ground state solutions for the nonlinear Schrödinge-Maxwell equations. J. Math. Anal. Appl. 345 (2008) 90-108.

[11] Chen, S .J., Tang, C.-L.: High energy solutions for the superlinear Schrödinger-Maxwell equations. Nonlinear Anal. 71(2009) 4927-4934 .

[12] Li, Q., Su, H., Wei, Z.: Existence of infinitely many large solutions for the nonlinear Schrödinger-Maxwell equations. Nonlinear Anal. 72 (2010) 4264-4270.

[13] Sun, J: Infinitely many solutions for a class of sublinear Schrödinger-Maxwell equations. J. Math. Anal. Appl. 390 (2012) 514-522.

[14] Wen-nian Huang, X.H. Tang, The existence of infinitely many solutions for the nonlinear Schrödinger-Maxwell equations. Results. Math. 65(2014) 223-234.

[15] Willem, M.: Minimax Theorems. Birkhauser, Boston (1996).

[16] Zou, W.: Variant fountain theorems and their applications. Manuscripta Math. 104 (2001) 343-358.

[17] X. Chang, Ground state solutions of asymptotically linear fractional Schröinger equations. J Math Phys. 54 (2013) 061504.

[18] P. Felmer, A. Quaas, and J. G. Tan, Positive solutions of nonlinear Schrödinger equation with the fractional Laplacian. Proc. - R. Soc. Edinburgh, Sect. A: Math. 142 (2012) 1237-1262 .

[19] Zifei Shen and Fashun Gao, On the Existence of Solutions for the Critical Fractional Laplacian Equation in $\mathbb{R}^N$. Abstract and Applied Analysis, 2014, Article ID 143741, 10 pages.

[20] Hajaiej H, Yu X, Zhai Z. Fractional Gagliardo-Nirenberg and Hardy inequalities under Lorentz norms. J. Math. Anal. Appl. 396 (2012) 569-577.

[21] Elliott H. Lieb, Michael Loss, Analysis, Second edition (Graduate Studies in Mathematics 14)-AMS Bookstore (2001).

[22] Yosida, K.: Functional Analysis, 6th edn. Springer-Verlag, New York (1999).

[23] Bartolo, T., Benci, V., Fortunato, D.: Abstract critical point theorems and applications to some nonlinear problems with strong resonance at infinity. Nonlinear Anal. 7, 241-273 (1983).